Longest Processing Time rule for identical parallel machines scheduling revisited

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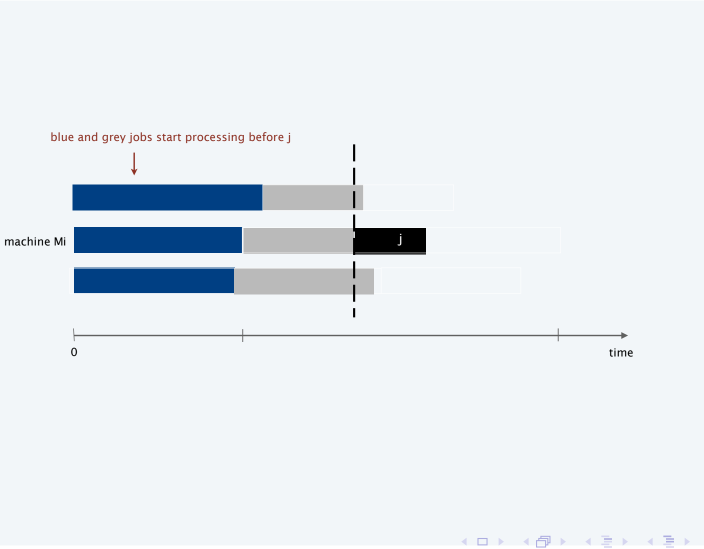
Outline

1. Introduction
2. *LPT* rule
3. *LPT* revisited
4. Improving the *LPT* bound
5. From approximation to heuristics
6. Computational testing

Introduction

- We consider problem $P_m||C_{max}$ where the goal is to schedule n jobs on m identical parallel machines M_i $(i = 1, \ldots, m)$ minimizing the makespan.



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- We employ Linear Programming to analyze the worst case performance of a simple modification of LPT that manages to improve the longstanding Graham's bound (from $\frac{4}{3} - \frac{1}{3m}$ to $\frac{4}{3} - \frac{1}{3(m-1)}$) for $m \geq 3$.

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- Then, we move from approximation to heuristics. By generalizing the proposed approach we obtain a simple $O(n \log n)$ procedure that drastically improves upon the performances of the LPT rule.
- On 780 benchmark literature instances (Iori and Martello 2008), the new procedure wins 513 times, ties 224 times and loses 43 times against LPT.

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- Denote the solution values of the *LPT* schedule and the optimal makespan by C_m^{LPT} and C_m^* respectively, where index m indicates the number of machines.
- Denote by $r_k = \frac{C_m^{LPT}}{C_m^*}$ the approximation ratio of the *LPT* schedule with k jobs assigned to the machine yielding the maximum completion time (the critical machine)

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- For each job i assigned by LPT in position j on a machine:

$$p_i \leq \frac{C_m^*}{j} \text{ - [Chen 1993].}$$

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Notice that

- $r_2 = 1$ for $m = 2$;
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$\Longrightarrow$ **Improving r_3 improves LPT.**
We concentrate then on instances where the critical job is in position 3.

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- $C^*_m = 3m, \qquad C^{LPT}_m = 4m - 1, \qquad r_3 = \frac{4m-1}{3m} = \frac{4}{3} - \frac{1}{3m}$.
- Worst-case always occurs with $2m + 1 = n$ jobs where the critical job is job $2m + 1 = n$ in position 3 and when $C^*_m = \sum_{i=1}^{n} p_i/m$.

LPT revisited

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- We employ **Linear Programming** to perform the analysis.

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Proposition

If LPT schedules at least 3 jobs on a non crit. machine before assigning the crit. job, then LPT has an approx. bound $\leq \frac{4}{3} - \frac{1}{3(m-1)}$ for $m \geq 5$.

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- As n is in position 3, condition $p_n \leq \frac{C_m^*}{3}$ holds.

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- We associate non-negative variables π and ξ with p_n and $\sum_{j=1}^{n} p_j$.
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- The following LP model is implied:

$$
\begin{aligned}
\text{minimize} \quad & opt && (1)\\
\text{subject to} \quad & -m \cdot opt + \xi \le 0 && (2)\\
& 3 \cdot \pi - c_1 \le 0 && (3)\\
& l - c_1 \le 0 && (4)\\
& (m-2)l - c_2 \le 0 && (5)\\
& c_1 + l + \pi + c_2 - \xi = 0 && (6)\\
& l + \pi = 1 && (7)\\
& \pi - \frac{opt}{3} \le 0 && (8)\\
& \pi, \xi, c_1, c_2, l, opt \ge 0 && (9)
\end{aligned}
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$$\pi = \frac{m-1}{4m-5}; \qquad \xi = \frac{3m(m-1)}{4m-5};$$

$$c_1 = \frac{3(m-1)}{4m-5}; \qquad c_2 = \frac{(m-2)(3(m-1)-1)}{4m-5};$$

$$l = \frac{3(m-1)-1}{4m-5}; \qquad opt = \frac{3(m-1)}{4m-5}.$$

LPT revisited: $2m+2 \leq n \leq 3m$ (LP formulation)

- The proposed LP model is continuous and contains just 6 variables and 7 constraints for any fixed m.
- By strong duality (and a little bit of reverse engineering) it is possible to show that in the optimal solution, for any $m \geq 5$, the variables values are as follows

$$\pi = \frac{m-1}{4m-5}; \qquad \xi = \frac{3m(m-1)}{4m-5};$$
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- Correspondingly, for any $m \geq 5$, we have $\frac{C_m^{LPT}}{C_m^*} \leq 1/opt = \frac{4m-5}{3(m-1)} = \frac{4}{3} - \frac{1}{3(m-1)}$.

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- Notice that **this bound is not tight**.

LPT revisited: $2m+2 \leq n \leq 3m$

- A more general results, provided below, actually holds.

Proposition

If LPT schedules at least k jobs on a non crit. machine before assigning the crit. job, then LPT has an approx. bound $\leq \frac{k+1}{k} - \frac{1}{k(m-1)}$ for $m \geq k+2$.

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- For $m \leq 4$, by lp-modeling and partial enumeration it is possible to obtain the following result.

Proposition

In $P_m||C_{max}$ *instances with* $2m + 2 \leq n \leq 3m$, *LPT (with job n critical) has an approximation ratio* $\leq \frac{4}{3} - \frac{1}{3(m-1)}$ *for* $3 \leq m \leq 4$.

LPT revisited: further subcases

The following propositions also hold

Proposition

In $P_m||C_{max}$ instances with $n \leq 2m$ and $m \geq 3$, LPT has an approximation ratio $\leq \left(\frac{4}{3} - \frac{1}{3(m-1)}\right)$.

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In $P_m||C_{max}$ *instances with* $n \leq 2m$ *and* $m \geq 3$*,* LPT *has an approximation ratio* $\leq \left(\frac{4}{3} - \frac{1}{3(m-1)}\right)$.

Proposition

In $P_m||C_{max}$, $m \geq 3$ *and instances with* $n = 2m+1$*, if* LPT *loads at least three jobs on a machine before the critical job, then it has an approximation ratio* $\leq \left(\frac{4}{3} - \frac{1}{3(m-1)}\right)$.

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- The only case remaining is then related to instances with $n = 2m + 1$ where LPT schedules job n only in third position and n is critical.

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We consider the following Algorithm 1.

Input: $P_m||C_{max}$ instance with n jobs and $m \geq 3$ machines.
- Apply LPT yielding a schedule with makespan z_1 and $k-1$ jobs on the critical machine before job n.
- Apply $LPT' = LPT(\{n\})$ with solution value z_2.
- Apply $LPT'' = LPT(\{(n-k+1), ..., n\})$ with solution value z_3.
- Return $\min\{z_1, z_2, z_3\}$.

In practice, this algorithm applies LPT first and then re-applies LPT *after* having loaded on a machine first either its critical job n alone or the tuple of k jobs $n-k+1, ..., n$.

Handling instances with n jobs and $j' = 2m + 1 \neq n$

We consider first the case where $j' \neq n$ and there are jobs processed after the critical job in LPT and one of such jobs is critical in either LPT' or LPT''.

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- it is sufficient to exploit the difference between $\sum_{j=1}^{j'} p_j$ and $\sum_{j=1}^{n} p_j$.

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Handling instances with $n = 2m + 1$ jobs and $j' = n = 2m + 1$

- Note that LPT must couple jobs $1, \ldots, m$ respectively with jobs $2m, \ldots, m + 1$ on the m machines before scheduling job $2m + 1$, or else LPT has an approximation ratio $\leq \left(\frac{4}{3} - \frac{1}{3(m-1)}\right)$.

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- Hence, the LPT schedule is as follows

$$
\begin{aligned}
&M_1 : p_1, p_{2m} \\
&M_2 : p_2, p_{2m-1} \\
&\ldots \\
&M_{m-1} : p_{m-1}, p_{m+2} \\
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where job $2m + 1$ will be assigned to the machine with the least processing time.

Handling instances with $n = 2m + 1$ jobs and $j' = n = 2m + 1$

We consider two specific cases:

1. $p_{2m+1} \geq p_1 - p_m$. $\Longrightarrow$ The LPT' schedule is as follows

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2. $p_{2m+1} < p_1 - p_m$.

Case $j' = n = 2m + 1$, $p_{2m+1} \geq p_1 - p_m$, LPT' makespan is on M_1

- If LPT' is not optimal, then $C^*_m \geq p_{m-1} + p_m$.

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If $p_{2m+1} \geq p_1 - p_m$ and LPT' makespan is equal to $p_{2m+1} + p_m + p_{2m}$, then the proposed algorithm has an approximation ratio not superior to $\frac{7}{6}$.

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- Proof: we again employ Linear Programming to evaluate the performance of LPT'. We consider non-negative variables x_j associated with p_j $(j = 1, \ldots, n)$ and a positive parameter $OPT > 0$ associated with C^*_m.

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The LP model.

$$
\begin{aligned}
\text{maximize} \quad & x_{(2m+1)} + x_m + x_{2m} && (10)\\
\text{subject to} \quad & x_{(m-1)} + x_m \leq OPT && (11)\\
& x_{(2m-1)} + x_{2m} + x_{(2m+1)} \leq OPT && (12)\\
& x_{(2m+1)} - (x_1 - x_m) \geq 0 && (13)\\
& x_1 - x_{(m-1)} \geq 0 && (14)\\
& x_{(m-1)} - x_m \geq 0 && (15)\\
& x_m - x_{(m+1)} \geq 0 && (16)\\
& x_{(m+1)} - x_{(2m-1)} \geq 0 && (17)\\
& x_{(2m-1)} - x_{2m} \geq 0 && (18)\\
& x_{2m} - x_{(2m+1)} \geq 0 && (19)\\
& x_1, x_{(m-1)}, x_m, x_{(m+1)}, x_{(2m-1)}, x_{2m}, x_{(2m+1)} \geq 0 && (20)
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- Constraints (14)–(19) state that the considered relevant jobs are sorted by non-increasing processing times.
- Constraints (20) indicate that the variables are non-negative.
- Further viable constraints where not necessary to reach the required result. By setting $OPT = 1$, the cost function has value $\frac{7}{6}$.

Further cases and subcases for $j' = n = 2m + 1$

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If $p_{2m+1} < p_1 - p_m$, LPT has a performance guarantee not superior to $\frac{15}{13}$ for $m = 3$ and $\frac{4}{3} - \frac{1}{2m-1}$ for $m \geq 4$.

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Putting things together, the following Theorem holds

Theorem

The proposed algorithm has an approximation ratio not superior to $\frac{4}{3} - \frac{1}{3(m-1)}$ for $m \geq 3$.

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- Putting things together, for $m = 2$, the approx. ratio of the proposed algorithm is not superior to $9/8$.

From approximation to heuristics

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- We propose then an alternative approach that splits the sorted job set in tuples of m consecutive jobs $(1, \ldots, m; m+1, \ldots, 2m;$ etc.$)$ and sorts the tuples in non-increasing order of the difference between the largest job and the smallest job in the tuple. Then a list scheduling is applied to the set of sorted tuples. We denote this approach as $SLACK$.

From approximation to heuristics

The $SLACK$ heuristic:

Input: $P_m||C_{max}$ instance m machines and n jobs with processing times p_j $(j = 1, \ldots, n)$.
- Sort items by non-increasing p_j.
- Consider $\left\lceil \frac{n}{m} \right\rceil$ tuples of size m given by jobs $1, \ldots, m; m+1, \ldots, 2m$, etc..
If n is not multiple of m, add dummy jobs with null proc. time in the last tuple.
- For each tuple, compute the associated slack, namely
$p_1 - p_m, p_{(m+1)} - p_{2m}, \ldots, p_{(n-m+1)} - p_n$.
- Sort tuples by non-increasing slack and then fill a list of consecutive jobs in the sorted tuples.
- Apply List Scheduling to this job ordering and return the solution.

From approximation to heuristics

The $SLACK$ heuristic:

Input: $P_m||C_{max}$ instance m machines and n jobs with processing times p_j $(j = 1, \ldots, n)$.
- Sort items by non-increasing p_j.
- Consider $\left\lceil \frac{n}{m} \right\rceil$ tuples of size m given by jobs $1, \ldots, m; m+1, \ldots, 2m$, etc..
If n is not multiple of m, add dummy jobs with null proc. time in the last tuple.
- For each tuple, compute the associated slack, namely
$p_1 - p_m, p_{(m+1)} - p_{2m}, \ldots, p_{(n-m+1)} - p_n$.
- Sort tuples by non-increasing slack and then fill a list of consecutive jobs in the sorted tuples.
- Apply List Scheduling to this job ordering and return the solution.

Since the construction and sorting of the tuples can be performed in $\mathcal{O}(m \log m)$, the running time of $SLACK$ is $\mathcal{O}(n \log n)$ due to the initial jobs LPT sorting.

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- In *uniform instances* the processing times are integer uniformly distributed in the range $[a, b]$. In *non-uniform instances*, 98% of the processing times are integer uniformly distributed in $[0.9(b-a), b]$ while the remaining ones are uniformly distributed in $[a, 0.2(b-a)]$. For both classes, we have $a = 1; b = 100, 1000, 10000$.

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- For each class, the following values were considered for the number of machines and jobs: $m = 5, 10, 25$ and $n = 10, 50, 100, 500, 1000$.
- For each pair (m, n) with $m < n$, 10 instances were generated for a total of 780 instances.

			SLACK wins		draws		*LPT* wins	
$[a, b]$	m	Instances	#	(%)	#	(%)	#	(%)
1-100	5	50	31	(62.0)	16	(32.0)	3	(6.0)
	10	40	32	(80.0)	8	(20.0)	0	(0.0)
	25	40	23	(57.5)	17	(42.5)	0	(0.0)
1-1000	5	50	39	(78.0)	10	(20.0)	1	(2.0)
	10	40	40	(100.0)	0	(0.0)	0	(0.0)
	25	40	27	(67.5)	12	(30.0)	1	(2.5)
1-10000	5	50	39	(78.0)	10	(20.0)	1	(2.0)
	10	40	40	(100.0)	0	(0.0)	0	(0.0)
	25	40	28	(70.0)	10	(25.0)	2	(5.0)

Table: $P_m||C_{max}$ non uniform instances.

Computational testing

			SLACK wins		draws		*LPT* wins	
$[a, b]$	m	Instances	#	(%)	#	(%)	#	(%)
1-100	5	50	12	(24.0)	37	(74.0)	1	(2.0)
	10	40	14	(35.0)	20	(50.0)	6	(15.0)
	25	40	10	(25.0)	29	(72.5)	1	(2.5)
1-1000	5	50	32	(64.0)	15	(30.0)	3	(6.0)
	10	40	27	(67.5)	5	(12.5)	8	(20.0)
	25	40	24	(60.0)	12	(30.0)	4	(10.0)
1-10000	5	50	36	(72.0)	12	(24.0)	2	(4.0)
	10	40	37	(92.5)	0	(0.0)	3	(7.5)
	25	40	22	(55.0)	11	(27.5)	7	(17.5)

Table: $P_m||C_{max}$ uniform instances.

Computational testing

- $SLACK$ shows up to be clearly superior to LPT: on 780 benchmark literature instances, $SLACK$ wins 513 times, ties 224 times and loses 43 times only.

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- $SLACK$ shows up to be clearly superior to LPT: on 780 benchmark literature instances, $SLACK$ wins 513 times, ties 224 times and loses 43 times only.
- If LPT'' is added to SLACK, then $SLACK+LPT''$ compared to LPT wins 529 times, ties 213 times and loses 38 times only.