The Longest Processing Time rule for identical parallel machines revisited

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AFIA / ROADEF - September 10, 2020

Outline

ILP modeling and approximation

- Every standard undergraduate course on Operations Research (OR) embeds a section devoted to [Integer] Linear Programming (ILP) Modeling.

- OR experts and practitioners apply ILP models in order to
  - provide formal representations of real problems;
  - directly compute the corresponding solution by means of ILP solvers (unfortunately does not always work that well...);
  - compute heuristic solutions by means of matheuristics procedures embedding the solutions of ILP subproblems ed into local search approaches;
  - **Derive approximation bounds** on problems where the related ILP formulations presents strong structural properties

Approximation algorithms: standard notation

- OPT = optimal solution value
- A = solution value of the approximation algorithm
- $r_A = \frac{A}{OPT}$ = approximation ratio.
- We are typically interested in approximation algorithms requiring **polynomial time** complexity.

Standard approximation via **problem dependent ILP modeling**: Vertex Cover

- Input: A graph $G = (V, E)$
- Definition: A vertex cover of G is a subset of V that covers (i.e., "touches") every edge in E.
- ILP formulation of the minimum vertex cover (MVC) problem

$$
\begin{aligned}
\text{MVC} &= \left\{ \begin{array}{ll} \min & \sum_{i \in V} x_i \\ & x_i + x_j \geqslant 1 \ \ \forall (i,j) \in E \\ & x_i \in \{0,1\} \ \ \forall i \in V \end{array} \right. \\
\text{MVC-}\textsc{r} &= \left\{ \begin{array}{ll} \min & \sum_{i \in V} x_i \\ & x_i + x_j \geqslant 1 \ \ \forall (i,j) \in E \\ & 0 \leq x_i \leq 1 \ \ \forall i \in V \end{array} \right.
\end{aligned}
$$

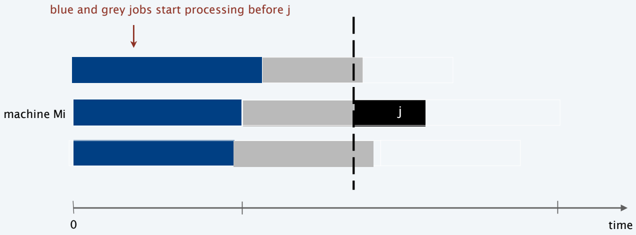
- Solving to optimality MVC-R (requires polynomial time) and setting $x_i = 1$ for all variables with value ≥ 0.5 provides a 2-approximation ratio.

Approximation via **non standard ILP modeling**

- We focus here on **non standard ILP modeling** for approximation.
- The aim is to mimick by ILP modeling the behavior of a procedure (typically greedy).
- We apply this approach to
  - Machine Scheduling: problem $P||C_{\max}$ and the LPT rule.

Parallel machines scheduling: Introduction

- We consider problem $P_m||C_{max}$ where the goal is to schedule n jobs on m identical parallel machines M_i $(i = 1, \ldots, m)$ minimizing the makespan.



Parallel machines scheduling: Introduction

- We consider problem $P_m||C_{max}$ where the goal is to schedule n jobs on m identical parallel machines M_i $(i = 1, \ldots, m)$ minimizing the makespan.
- We revisit the famous Longest Processing Time (LPT) rule proposed by Graham - 1969.
- LPT rule: sort the jobs $1, ..., n$ in non-ascending order of their processing times p_j $(j = 1, \ldots, n)$ and then assign one job at a time to the machine whose load is smallest so far.
- Assume the jobs indexed by non-increasing p_j $(p_j \geq p_{j+1},\ j = 1, \ldots, n-1)$.
- Denote the solution values of the LPT schedule and the optimal makespan by C_m^{LPT} and C_m^* respectively, where index m indicates the number of machines.
- Denote by $r_k = \frac{C_m^{LPT}}{C_m^*}$ the approximation ratio of the LPT schedule when k jobs are assigned to the machine yielding the maximum completion time (the critical machine)

$P_m||C_{max}$ problem and LPT rule properties

- $C_m^* \geq p_1$.
- $C_m^* \geq \frac{\sum_{j=1}^{n} p_j}{m}$.
- $C_m^{LPT} = C_m^*$ if $p_{j'} > \frac{C_m^*}{3}$ (j' denotes the critical job).
- $C_m^{LPT} \leq \frac{\sum_{j=1}^{j'-1} p_j}{m} + p_{j'} \leq C_m^* + p_{j'}(1 - \frac{1}{m}) \leq (\frac{4}{3} - \frac{1}{3m})C_m^*$ - [Graham 1969].
- For each job i assigned by LPT in position j on a machine: $p_i \leq \frac{C_m^*}{j}$ - [Chen 1993].

LPT rule properties:

Known LPT approximation ratios.

- $r_1 = 1$.
- $r_2 = \frac{4}{3} - \frac{1}{3(m-1)}$ - [Chen 1993].
- $r_3 = \frac{4}{3} - \frac{1}{3m}$ - [Graham 1969].
- $r_k = \frac{k+1}{k} - \frac{1}{km} \quad k \geq 3$ [Coffman and Sethi 1976 - generalizes Graham].

Notice that

- $r_2 = 1$ for $m = 2$;
- $r_2 = r_4$ for $m = 3$, $r_2 < r_4$ for $m \geq 4$;
- $r_k < r_{k+1}$ for $k \geq 3$

$\Longrightarrow$ **Improving r_3 improves LPT.**
We concentrate then on instances where the critical job is in position 3.

Tight worst-case examples for LPT

- 2 machines - 5 jobs $\longrightarrow [3, 3, 2, 2, 2]$.
- $C^*_{m=2} = 6, \qquad C^{LPT}_{m=2} = 7, \qquad r_3 = \frac{4}{3} - \frac{1}{3m} = \frac{7}{6}$.
- 3 machines, 7 jobs $\longrightarrow [5, 5, 4, 4, 3, 3, 3]$.
- $C^*_{m=3} = 9, \qquad C^{LPT}_{m=3} = 11, \qquad r_3 = \frac{4}{3} - \frac{1}{3m} = \frac{11}{9}$.
- m machines, $2m + 1$ jobs
  $\longrightarrow [2m-1, 2m-1, 2m-2, 2m-2, ..., m, m, m]$.
- $C^*_m = 3m = \frac{\sum_{i=1}^{n} p_i}{m}, \qquad C^{LPT}_m = 4m - 1,$
  $r_3 = \frac{4m-1}{3m} = \frac{4}{3} - \frac{1}{3m}$.
- Worst-case always occurs with $2m + 1 = n$ jobs where the critical job is job $2m + 1 = n$ in position 3 and when $C^*_m = \sum_{i=1}^{n} p_i/m$.

LPT revisited

- We assume that the *LPT* critical job is the last one, namely $j' = n$. If not, we would have further jobs after the critical job that do not affect the makespan provided by *LPT* but can contribute to increase the optimal solution value.

- We analyze for $m \geq 3$:
  - $2m + 2 \leq j' = n \leq 3m$ (or else the critical job would be in position ≥ 4);
  - $j' = n = 2m + 1$.

- We employ **Linear Programming** to perform the analysis.

LPT revisited: $2m + 2 \leq n \leq 3m$

Proposition

If LPT schedules at least 3 jobs on a non crit. machine before assigning the crit. job, then LPT has an approx. bound $\leq \frac{4}{3} - \frac{1}{3(m-1)}$ for $m \geq 5$.

Sketch of proof.

- We assume n in position 3, or else either r_2 holds or at least r_4 holds. Hence, LPT schedules at least another job in position ≥ 3.
- We consider an LP model where we arbitrarily set the value C_m^{LPT} to 1 and minimize the value of C_m^*.
- L denotes the starting time of job n, i.e. $C_m^{LPT} = L + p_n$.
- C_1 is the compl. time of the non-crit. machine processing at least 3 jobs.
- C_2 is the sum of compl. times of the other $(m-2)$ machines, i.e. $C_2 = \sum_{j=1}^{n} p_j - C_1 - (L + p_n)$.
- Due to list scheduling, condition $\frac{C_2}{m-2} \geq L$ holds.
- As n is in position 3, condition $p_n \leq \frac{C_m^*}{3}$ holds.

LPT revisited: $2m + 2 \leq n \leq 3m$ (LP formulation)

- We associate non-negative variables p_n and $sump$ with p_n and $\sum_{j=1}^{n} p_j$.
- We associate non-negative variables c_1, c_2, l, opt with C_1, C_2, L and C_m^*.
- The following LP model (for given m) is implied:

$$\begin{aligned}
\text{minimize} \quad & opt && (1)\\
\text{subject to} \quad & -m \cdot opt + sump \leq 0 && (2)\\
& 3 \cdot p_n - c_1 \leq 0 && (3)\\
& l - c_1 \leq 0 && (4)\\
& (m-2)l - c_2 \leq 0 && (5)\\
& c_1 + l + p_n + c_2 - sump = 0 && (6)\\
& l + p_n = 1 && (7)\\
& p_n - \frac{opt}{3} \leq 0 && (8)\\
& p_n, sump, c_1, c_2, l, opt \geq 0 && (9)
\end{aligned}$$

LPT revisited: $2m+2 \leq n \leq 3m$ (LP formulation)

- The minimization of the objective function (1), after setting w.l.o.g LPT solution value to 1 (constraint (7)), provides an upper bound on the performance ratio of LPT rule.
- Constraint $(2: \; -m \cdot opt + sump \leq 0)$ represents the bound $C_m^* \geq \frac{\sum_{j=1}^{n} p_j}{m}$.
- Constraint $(3: \; 3 \cdot p_n - c_1 \leq 0)$ states that the value of c_1 is at the least $3p_n$, since 3 jobs with proc. time $\geq p_n$ are assigned to a non critical machine.
- Constraint $(4: \; l - c_1 \leq 0)$ states that the proc. time of the critical machine before the last job is loaded is less than the compl. time of the other machine processing at least three jobs.
- Constraint $(5: \; (m-2)l - c_2 \leq 0)$ fulfills the list scheduling requirement.
- Constraint $(6: \; c_1 + l + p_n + c_2 - sump = 0)$ guarantees that variable $sump$ represents $\sum_{j=1}^{n} p_j$
- Constraint (8) represents condition $p_n \leq \frac{C_m^*}{3}$.
- Constraints (9) state that all variables are non-negative.

LPT revisited: $2m+2 \leq n \leq 3m$ (LP formulation)

- The proposed LP model is continuous and contains just 6 variables and 7 constraints for any fixed m.
- By strong duality (and a little bit of reverse engineering) it is possible to show that in the optimal solution, for any $m \geq 5$, the variables values are as follows

$$p_n = \frac{m-1}{4m-5}; \qquad sump = \frac{3m(m-1)}{4m-5};$$

$$c_1 = \frac{3(m-1)}{4m-5}; \qquad c_2 = \frac{(m-2)(3(m-1)-1)}{4m-5};$$

$$l = \frac{3(m-1)-1}{4m-5}; \qquad opt = \frac{3(m-1)}{4m-5}.$$

- Correspondingly, for any $m \geq 5$, we have $\frac{C_m^{LPT}}{C_m^*} \leq 1/opt = \frac{4m-5}{3(m-1)} = \frac{4}{3} - \frac{1}{3(m-1)}$.
- Notice that **this bound is not tight**.

LPT revisited: $2m + 2 \leq n \leq 3m$

- A more general results, provided below, actually holds.

Proposition

If LPT schedules at least k jobs on a non crit. machine before assigning the crit. job, then LPT has an approx. bound $\leq \frac{k+1}{k} - \frac{1}{k(m-1)}$ for $m \geq k + 2$.

- For $3 \leq m \leq 4$, by lp-modeling and partial enumeration it is possible to obtain the following result.

Proposition

In $P_m||C_{max}$ instances with $2m + 2 \leq n \leq 3m$, LPT (with job n critical) has an approximation ratio $\leq \frac{4}{3} - \frac{1}{3(m-1)}$ for $3 \leq m \leq 4$.

LPT revisited: further subcases

The following propositions also hold

Proposition

In $P_m||C_{max}$ *instances with* $n \leq 2m$ *and* $m \geq 3$*, LPT has an approximation ratio* $\leq \left(\frac{4}{3} - \frac{1}{3(m-1)}\right)$.

Proposition

In $P_m||C_{max}$, $m \geq 3$ *and instances with* $n = 2m+1$*, if LPT loads at least three jobs on a machine before the critical job, then it has an approximation ratio* $\leq \left(\frac{4}{3} - \frac{1}{3(m-1)}\right)$.

- The only case remaining is then related to instances with $n = 2m+1$ where *LPT* schedules job n only in third position and n is critical.

Improving LPT for $n = 2m + 1$

- We consider a slight algorithmic variation where a set of the sorted jobs is first loaded on a machine and then LPT is applied on the remaining job set.
- We denote this variant as $LPT(\mathcal{S})$ where $\mathcal{S}$ represents the set of jobs assigned all together to a machine first.

We consider the following Algorithm 1.

Input: $P_m||C_{max}$ instance with n jobs and $m \geq 3$ machines.
- Apply LPT yielding a schedule with makespan z_1 and $k - 1$ jobs on the critical machine before job n.
- Apply $LPT' = LPT(\{n\})$ with solution value z_2.
- Apply $LPT'' = LPT(\{(n - k + 1), ..., n\})$ with solution value z_3.
- Return $\min\{z_1, z_2, z_3\}$.

In practice, this algorithm applies LPT first and then re-applies LPT *after* having loaded on a machine first either its critical job n alone or the tuple of k jobs $n - k + 1, ..., n$.

Handling instances with n jobs and $j' = 2m + 1 \neq n$

We consider first the case where $j' \neq n$ and there are jobs processed after the critical job in LPT and one of such jobs is critical in either LPT' or LPT''.

Proposition

In $P_m||C_{max}$ instances where there are jobs processed after the critical job in the LPT solution and one of such jobs (say job l) is critical in either LPT' or LPT'' schedules, Algorithm 1 has a performance guarantee of $\frac{4}{3} - \frac{7m-4}{3(3m^2+m-1)}$.

Proof hints (formal proof needs some more algebra):

- it is sufficient to exploit the difference between $\sum_{j=1}^{j'} p_j$ and $\sum_{j=1}^{n} p_j$.
- If $\sum_{j=j'+1}^{n} p_j$ is large enough, then $\frac{\sum_{j=1}^{j'} p_j}{m} + p_{j'}/m \ll \frac{\sum_{j=1}^{n} p_j}{m} + p_l/m$, namely, the bound on the LPT approx. ratio becomes small enough;
- if $\sum_{j=j'+1}^{n} p_j$ is small enough, then the approx. ratio of LPT' or LPT'' also becomes small enough.

Handling instances with $n = 2m + 1$ jobs and $j' = n = 2m + 1$

- Note that LPT must couple jobs $1, \ldots, m$ respectively with jobs $2m, \ldots, m+1$ on the m machines before scheduling job $2m+1$, or else LPT has an approximation ratio $\leq \left(\frac{4}{3} - \frac{1}{3(m-1)}\right)$.
- Hence, the LPT schedule is as follows

$$
\begin{aligned}
&M_1 : p_1, p_{2m} \\
&M_2 : p_2, p_{2m-1} \\
&\ldots \\
&M_{m-1} : p_{m-1}, p_{m+2} \\
&M_m : p_m, p_{m+1}
\end{aligned}
$$

where job $2m+1$ will be assigned to the machine with the least completion time.

Handling instances with $n = 2m + 1$ jobs and $j' = n = 2m + 1$

We consider two specific cases:

1. $p_{2m+1} \geq p_1 - p_m$. $\Longrightarrow$ The LPT' schedule is as follows

$$
\begin{aligned}
M_1 &: p_{2m+1}, p_m, p_{2m} \\
M_2 &: p_1, p_{2m-1} \\
M_3 &: p_2, p_{2m-2} \\
\dots \\
M_{m-1} &: p_{m-2}, p_{m+2} \\
M_m &: p_{m-1}, p_{m+1}
\end{aligned}
$$

   with subcases

   1. The LPT' makespan is on M_1.
   2. The LPT' makespan is on M_2,...M_m.

2. $p_{2m+1} < p_1 - p_m$.

Case $j' = n = 2m+1$, $p_{2m+1} \geq p_1 - p_m$, LPT' makespan is on M_1

- If LPT' is not optimal, then $C_m^* \geq p_{m-1} + p_m$.
- We get the following result.

Proposition

If $p_{2m+1} \geq p_1 - p_m$ and LPT' makespan is equal to $p_{2m+1} + p_m + p_{2m}$, then the proposed algorithm has an approximation ratio not superior to $\frac{7}{6}$.

- Proof: we again employ Linear Programming to evaluate the performance of LPT'. We consider non-negative variables x_j associated with p_j ($j = 1, \ldots, n$) and a positive parameter $OPT > 0$ associated with C_m^*.

Case $j' = n = 2m+1$, $p_{2m+1} \geq p_1 - p_m$, LPT' makespan is on M_1

The LP model.

$$
\begin{aligned}
\text{maximize} \quad & x_{(2m+1)} + x_m + x_{2m} && (10)\\
\text{subject to} \quad & x_{(m-1)} + x_m \leq OPT && (11)\\
& x_{(2m-1)} + x_{2m} + x_{(2m+1)} \leq OPT && (12)\\
& x_{(2m+1)} - (x_1 - x_m) \geq 0 && (13)\\
& x_1 - x_{(m-1)} \geq 0 && (14)\\
& x_{(m-1)} - x_m \geq 0 && (15)\\
& x_m - x_{(m+1)} \geq 0 && (16)\\
& x_{(m+1)} - x_{(2m-1)} \geq 0 && (17)\\
& x_{(2m-1)} - x_{2m} \geq 0 && (18)\\
& x_{2m} - x_{(2m+1)} \geq 0 && (19)\\
& x_1, x_{(m-1)}, x_m, x_{(m+1)}, x_{(2m-1)}, x_{2m}, x_{(2m+1)} \geq 0 && (20)
\end{aligned}
$$

Case $j' = n = 2m + 1$, $p_{2m+1} \geq p_1 - p_m$, LPT' makespan is on M_1

- The objective function value (10) represents an upper bound on the worst case performance of the algorithm.
- Constraints (11)–(12) correspond to $C^*_m \geq p_{m-1} + p_m$ and $C^*_m \geq p_{2m-1} + p_{2m} + p_{2m+1}$.
- Constraint (13) simply represents the initial assumption $p_{2m+1} \geq p_1 - p_m$.
- Constraints (14)–(19) state that the considered relevant jobs are sorted by non-increasing processing times.
- Constraints (20) indicate that the variables are non-negative.
- Further viable constraints where not necessary to reach the required result. By setting $OPT = 1$, the cost function has value $\frac{7}{6}$.

Further cases and subcases for $j' = n = 2m + 1$

By means of further LP models, the following results hold

Proposition

If $p_{2m+1} \geq p_1 - p_m$ and LPT' makespan is on $M_2, ..., M_m$, then LPT' has a performance guarantee of $\frac{15}{13}$ for $m = 3$ and $\frac{4}{3} - \frac{1}{2m-1}$ for $m \geq 4$.

Proposition

If $p_{2m+1} < p_1 - p_m$, LPT has a performance guarantee not superior to $\frac{15}{13}$ for $m = 3$ and $\frac{4}{3} - \frac{1}{2m-1}$ for $m \geq 4$.

Putting things together, the following Theorem holds

Theorem

The proposed algorithm has an approximation ratio not superior to $\frac{4}{3} - \frac{1}{3(m-1)}$ for $m \geq 3$.

From approximation to heuristics

- W.r.t. the worst-case analysis for $m \geq 3$, the relevant subcase was the one with $p_{2m+1} \geq p_1 - p_m$ and LPT' required to schedule p_{2m+1} first and then apply list scheduling first to the sorted jobset $p_1, ..., p_m$ according to LPT and then to the sorted jobset $p_{m+1}, ..., p_{2m}$ always according to LPT.

- We propose then an alternative approach that splits the sorted job set in tuples of m consecutive jobs $(1, \ldots, m; m+1, \ldots, 2m;$ etc.) and sorts the tuples in non-increasing order of the difference between the largest job and the smallest job in the tuple. Then a list scheduling is applied to the set of sorted tuples. We denote this approach as $SLACK$.

From approximation to heuristics

The $SLACK$ heuristic:

Input: $P_m||C_{max}$ instance m machines and n jobs with processing times p_j $(j = 1, \ldots, n)$.
- Sort items by non-increasing p_j.
- Consider $\left\lceil \frac{n}{m} \right\rceil$ tuples of size m given by jobs $1, \ldots, m; m+1, \ldots, 2m$, etc..
If n is not multiple of m, add dummy jobs with null proc. time in the last tuple.
- For each tuple, compute the associated slack, namely
$p_1 - p_m, p_{(m+1)} - p_{2m}, \ldots, p_{(n-m+1)} - p_n$.
- Sort tuples by non-increasing slack and then fill a list of consecutive jobs in the sorted tuples.
- Apply List Scheduling to this job ordering and return the solution.

Since the construction and sorting of the tuples can be performed in $\mathcal{O}(n + m \log m)$, the running time of $SLACK$ is $\mathcal{O}(n \log n)$ due to the initial jobs LPT sorting.

Computational testing

We compared $SLACK$ to LPT on benchmark literature instances (Iori, Martello 2008)

- Two classical classes of instances from literature are considered: *uniform instances* (França et al. 1994) and *non-uniform instances* (Frangioni et al. 2004).
- In *uniform instances* the processing times are integer uniformly distributed in the range $[a, b]$. In *non-uniform instances*, 98% of the processing times are integer uniformly distributed in $[0.9(b-a), b]$ while the remaining ones are uniformly distributed in $[a, 0.2(b-a)]$. For both classes, we have $a = 1; b = 100, 1000, 10000$.
- For each class, the following values were considered for the number of machines and jobs: $m = 5, 10, 25$ and $n = 10, 50, 100, 500, 1000$.
- For each pair (m, n) with $m < n$, 10 instances were generated for a total of 780 instances.

Computational testing

			SLACK wins		draws		*LPT* wins	
$[a, b]$	m	Instances	#	(%)	#	(%)	#	(%)
1-100	5	50	31	(62.0)	16	(32.0)	3	(6.0)
	10	40	32	(80.0)	8	(20.0)	0	(0.0)
	25	40	23	(57.5)	17	(42.5)	0	(0.0)
1-1000	5	50	39	(78.0)	10	(20.0)	1	(2.0)
	10	40	40	(100.0)	0	(0.0)	0	(0.0)
	25	40	27	(67.5)	12	(30.0)	1	(2.5)
1-10000	5	50	39	(78.0)	10	(20.0)	1	(2.0)
	10	40	40	(100.0)	0	(0.0)	0	(0.0)
	25	40	28	(70.0)	10	(25.0)	2	(5.0)

Table: $P_m||C_{max}$ non uniform instances.

Computational testing

			SLACK wins		draws		*LPT* wins	
$[a, b]$	m	Instances	#	(%)	#	(%)	#	(%)
1-100	5	50	12	(24.0)	37	(74.0)	1	(2.0)
	10	40	14	(35.0)	20	(50.0)	6	(15.0)
	25	40	10	(25.0)	29	(72.5)	1	(2.5)
1-1000	5	50	32	(64.0)	15	(30.0)	3	(6.0)
	10	40	27	(67.5)	5	(12.5)	8	(20.0)
	25	40	24	(60.0)	12	(30.0)	4	(10.0)
1-10000	5	50	36	(72.0)	12	(24.0)	2	(4.0)
	10	40	37	(92.5)	0	(0.0)	3	(7.5)
	25	40	22	(55.0)	11	(27.5)	7	(17.5)

Table: $P_m||C_{max}$ uniform instances.

Computational testing

- $SLACK$ shows up to be clearly superior to LPT: on 780 benchmark literature instances, $SLACK$ wins 513 times, ties 224 times and loses 43 times only.
- If LPT'' is added to SLACK, then $SLACK+LPT''$ compared to LPT wins 529 times, ties 213 times and loses 38 times only.

Conclusions

- We discussed how non standard ILP modeling can be successfully applied to derive improved approximation results.
- We considered problem $P_m||C_{max}$ and revisited the LPT rule.
- By means of Linear Programming we improved Graham's bound from $\frac{4}{3} - \frac{1}{3m}$ to $\frac{4}{3} - \frac{1}{3(m-1)}$ for $m \geq 3$.
- By similar analysis, a linear time algorithm for problem $P2||C_{\max}$ with a 13/12 approximation ratio can be derived;
- From the approximation analysis, we derived a simple $O(n \log n)$ heuristic procedure that drastically improves upon the performances of LPT.
- We believe that the proposed LP-based analysis can be successfully applied in approximation theory as a valid alternative to formal proof systems based on analytical derivation.

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